# Static black holes of metric–affine gravity in the presence of matter

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## Abstract

We investigate spherically symmetric and static gravitational fields representing black hole configurations in the framework of metric–affine gauge theories of gravity (MAG) in the presence of different matter fields. It is shown that in the *triplet ansatz* sector of MAG, black hole configurations in the presence of non–Abelian matter fields allow the existence of black hole hair. We analyze several cases of matter fields characterized by the presence of hair and for all of them we show the validity of the no short hair conjecture.

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## I. INTRODUCTION

One of the first attempts to consider a non–Riemannian description of the gravitational field coupled to an electromagnetic field is due to Weyl [1]. Although this approach is considered today as unsuccessful, recent developments through a theory that unifies all the fundamental interactions are reviving the interest in non–Riemannian structures. For instance, the unification scheme in the framework of string theory indicates that the classical Riemannian description is not valid on all scales.

Indeed, the theory of the quantum superstring [2] indicates that non–Riemannian features are present on the scale of the Planck length. It turns out that low–energy dilaton and axi–dilaton interactions are tractable in terms of a connection that leads to a non–Riemannian geometrical structure with a particular torsion and nonmetricity fields. Therefore, it is interesting to investigate gravity theories which generalize the pure Riemannian geometrical structure of Einstein's theory.

MAG is a gauge theory of the 4–dimensional affine group endowed with a metric. As a gauge theory, it finds its appropriate form if expressed with respect to arbitrary frames or *coframes*. The corresponding gravitational potentials are the metric $g_{\alpha\beta}$, the coframe $\vartheta^{\alpha}$, and the connection 1–form $\Gamma_{\alpha}{}^{\beta}$, with values in the Lie algebra of the 4–dimensional linear group $GL(4,R)$. Therefore, spacetime is described by a metric–affine geometry with the gravitational *field strengths* nonmetricity $Q_{\alpha\beta} := -Dg_{\alpha\beta}$, torsion $T^{\alpha} := D\vartheta^{\alpha}$, and curvature $R_{\alpha}{}^{\beta} := d\Gamma_{\alpha}{}^{\beta} - \Gamma_{\alpha}{}^{\gamma} \wedge \Gamma_{\gamma}{}^{\beta}$. Thus, the post–Riemannian components, nonmetricity and torsion, are dynamical variables which together with the metric and the connection provide an alternative description of gravitational physics.

In this work, we consider MAG as a gravity theory and investigate static spherically symmetric fields describing black hole configurations. In particular, we investigate black holes of the triplet ansatz sector of MAG in the presence of matter represented by a $SU(2)$ Yang–Mills, Skyrme, Yang–Mills–dilaton, Yang–Mills–Higgs and a non–Abelian Proca field. We first show that, in general, the presence of matter with non–Abelian structure does not

allow to apply the arguments used to prove no–hair theorems. We show that in all these cases black hole hair exists and it must extend beyond a surface situated at 3/2 the horizon radius. i.e., we prove the validity of the *no short hair conjecture* in this sector of MAG.

In Section II we review the main aspects of MAG and its *triplet ansatz* sector together with Obukhov's equivalence theorem. Besides the pure geometric components of MAG, we consider an additional matter field. In Section III we analyze the general equations of motion for non–Abelian matter fields, consider a static spherically symmetric configuration under the assumption that it represents the gravitational field of a black hole, and prove the validity of the *no short hair conjecture*. Finally, in Section IV we discuss our results.

## II. FIELD EQUATIONS OF MAG AND THE TRIPLET ANSATZ

Let us consider a frame field and a coframe field denoted by

$$e_\alpha = e^\mu{}_\alpha\, \partial_\mu \,, \qquad\qquad \vartheta^\beta = e_\mu{}^\beta dx^\mu \,, \tag{2.1}$$

respectively. The $GL(4,R)$–covariant derivative for a tensor valued $p$–form is

$$D = d + \Gamma_\alpha{}^\beta\ \rho(L^\alpha{}_\beta) \wedge \,, \tag{2.2}$$

where $\rho(L^\alpha{}_\beta)$ is the representation of $GL(4,R)$ and $L^\alpha{}_\beta$ are the generators; the connection one–form is $\Gamma_\alpha{}^\beta = \Gamma_{\mu\alpha}{}^\beta dx^\mu$. The nonmetricity one–form, the torsion and curvature two–forms read

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} \,, \quad T^\alpha := D\vartheta^\alpha \,, \quad R_\alpha{}^\beta := d\Gamma_\alpha{}^\beta - \Gamma_\alpha{}^\gamma \wedge \Gamma_\gamma{}^\beta \,, \tag{2.3}$$

respectively, and the Bianchi identities are

$$DQ_{\alpha\beta} \equiv 2R_{(\alpha\beta)} \,, \quad DT^\alpha \equiv R_\gamma{}^\alpha \wedge \vartheta^\gamma \,, \quad DR_\alpha{}^\beta \equiv 0 \,. \tag{2.4}$$

It is worthwhile to stress the fact that $Q_{\alpha\beta}$, $T^\alpha$ and $R_\alpha{}^\beta$ play the role of field strengths.

We will consider a metric–affine theory described by the particular Lagrangian

$$\mathcal{L} = V_{\rm MAG} + \mathcal{L}_{\rm MAT}\,, \tag{2.5}$$

where $\mathcal{L}_{\rm MAT}$ represents the Lagrangian of the matter field.

In a metric–affine spacetime, the curvature has *eleven* irreducible pieces [3], whereas the nonmetricity has *four* and the torsion *three* irreducible pieces. The most general parity conserving Lagrangian $V_{\rm MAG}$ which has been constructed in terms of all irreducible pieces of the post–Riemannian components has been investigated previously [4] and reads:

$$\begin{aligned} V_{\rm MAG} = \frac{1}{2\kappa}\, &\Bigg[-a_0\, R^{\alpha\beta}\wedge\eta_{\alpha\beta} - 2\lambda\,\eta + T^\alpha\wedge{}^*\left(\sum_{I=1}^{3} a_I\,{}^{(I)}T_\alpha\right) \\ &+\, 2\left(\sum_{I=2}^{4} c_I\,{}^{(I)}Q_{\alpha\beta}\right)\wedge\vartheta^\alpha\wedge{}^*T^\beta + Q_{\alpha\beta}\wedge{}^*\left(\sum_{I=1}^{4} b_I\,{}^{(I)}Q^{\alpha\beta}\right) \\ &+\, b_5\left({}^{(3)}Q_{\alpha\gamma}\wedge\vartheta^\alpha\right)\wedge{}^*\left({}^{(4)}Q^{\beta\gamma}\wedge\vartheta_\beta\right)\Bigg] \\ &-\,\frac{1}{2\rho}\,R^{\alpha\beta}\wedge{}^*\Bigg(\sum_{I=1}^{6} w_I\,{}^{(I)}W_{\alpha\beta} + w_7\,\vartheta_\alpha\wedge(e_\gamma\rfloor{}^{(5)}W^\gamma{}_\beta) \\ &+\,\sum_{I=1}^{5} z_I\,{}^{(I)}Z_{\alpha\beta} + z_6\,\vartheta_\gamma\wedge(e_\alpha\rfloor{}^{(2)}Z^\gamma{}_\beta) + \sum_{I=7}^{9} z_I\,\vartheta_\alpha\wedge(e_\gamma\rfloor{}^{(I-4)}Z^\gamma{}_\beta)\Bigg)\,. \end{aligned} \tag{2.6}$$

The Minkowski metric is $o_{\alpha\beta} = \mathrm{diag}(-+++)$, $^*$ is the Hodge dual, $\eta := {}^*1$ is the volume four–form, the constant $\lambda$ is the cosmological constant, $\rho$ the strong gravity coupling constant, the constants $a_0, \ldots a_3,\ b_1, \ldots b_5,\ c_2, c_3, c_4,\ w_1, \ldots w_7,\ z_1, \ldots z_9$ are dimensionless. We have introduced in the curvature square term the irreducible pieces of the antisymmetric part $W_{\alpha\beta} := R_{[\alpha\beta]}$ and the symmetric part $Z_{\alpha\beta} := R_{(\alpha\beta)}$ of the curvature 2–form. In $Z_{\alpha\beta}$, we have the purely *post*–Riemannian part of the curvature. Note the peculiar cross terms with $c_I$ and $b_5$.

We will consider here only the simplest non–trivial case of torsion and nonmetricity with shear. Then, for the nonmetricity we use the ansatz

$$Q_{\alpha\beta} = {}^{(3)}Q_{\alpha\beta} + {}^{(4)}Q_{\alpha\beta}\,, \tag{2.7}$$

where

$${}^{(3)}Q_{\alpha\beta} = \frac{4}{9}\left(\vartheta_{(\alpha}e_{\beta)}\rfloor\Lambda - \frac{1}{4}g_{\alpha\beta}\Lambda\right)\,, \qquad \text{with} \qquad \Lambda := \vartheta^\alpha e^\beta\rfloor\not{Q}_{\alpha\beta}\,, \tag{2.8}$$

is the proper shear piece and ${}^{(4)}Q_{\alpha\beta} = Q\, g_{\alpha\beta}$ represents the dilation piece, where $Q := (1/4)\, Q_{\gamma}{}^{\gamma}$ is the Weyl one–form, and $\not{Q}_{\alpha\beta} := Q_{\alpha\beta} - Q\, g_{\alpha\beta}$ is the traceless piece of the nonmetricity. Other pieces of the irreducible decomposition of the nonmetricity [3] are taken to be zero.

Let us choose for the torsion only the covector piece as non–vanishing:

$$T^{\alpha} = {}^{(2)}T^{\alpha} = \frac{1}{3}\,\vartheta^{\alpha} \wedge T\,, \qquad \text{with} \qquad T := e_{\alpha}\rfloor T^{\alpha}\,. \tag{2.9}$$

Thus we are left with a triplet of non–trivial one–forms $Q$, $\Lambda$, and $T$ for which we make the following ansatz

$$Q = \frac{k_0}{k_1}\Lambda = \frac{k_0}{k_2}T\,, \tag{2.10}$$

where $k_0$, $k_1$ and $k_2$ are given in terms of the gravitational coupling constants (for details, see [4]). This is the so–called *triplet ansatz* sector of MAG theories [5,6,8].

Consequently, here, we limit ourselves to the special case in which the only surviving strong gravity piece is the square of the segmental curvature (with vanishing cosmological constant), i.e.

$$\begin{aligned} V_{MAG} = \frac{1}{2\kappa}\, \Big[ &-a_0\, R^{\alpha\beta} \wedge \eta_{\alpha\beta} + a_2\, T^{\alpha} \wedge {}^{*(2)}T_{\alpha} \\ &+ 2\left(c_3\, {}^{(3)}Q_{\alpha\beta} + c_4\, {}^{(4)}Q_{\alpha\beta}\right) \wedge \vartheta^{\alpha} \wedge {}^{*}T^{\beta} \\ &+ Q_{\alpha\beta} \wedge {}^{*}\!\left(b_3\, {}^{(3)}Q^{\alpha\beta} + b_4\, {}^{(4)}Q^{\alpha\beta}\right)\Big] \\ &- \frac{z_4}{2\rho}\, R^{\alpha\beta} \wedge {}^{*(4)}Z_{\alpha\beta}\,, \end{aligned} \tag{2.11}$$

where

$$-\frac{z_4}{2\rho}\, R_{\alpha}{}^{\alpha} \wedge {}^{*}Z_{\beta}{}^{\beta} = -\frac{2z_4}{\rho}\, dQ \wedge {}^{*}dQ \tag{2.12}$$

is the kinetic term for the Weyl one–form.

Under the above given assumptions it is now straightforward to apply Obukhov's equivalence theorem [5–7] according to which the field equations following from the pure geometrical part of the Lagrangian (2.5), i.e., $V_{MAG}$, are equivalent to Einstein's equations with

an energy–momentum tensor determined by a Proca field. In the case investigated here we have an additional term due to the presence of the matter field in (2.5). Thus, the field equations read

$$\frac{a_0}{2}\,\eta_{\alpha\beta\gamma}\wedge\tilde{R}^{\beta\gamma}=\kappa\,\Sigma_\alpha\,, \tag{2.13}$$

$$d\,{}^*H+m^2\,{}^*\phi=0, \tag{2.14}$$

where $\phi$ represents the Proca 1–form, $H\equiv d\phi$, $m$ is completely given in terms of the coupling constants, and a tilde denotes the Riemannian part of the curvature. The energy–momentum current entering the right hand side of the Einstein equations is given by

$$\Sigma_\alpha=\Sigma_\alpha^{(\phi)}+\Sigma_\alpha^{(\mathrm{MAT})}\,, \tag{2.15}$$

where

$$\begin{aligned}\Sigma_\alpha^{(\phi)}:=\tfrac{z_4k_0^2}{2\rho}\,\{&(e_\alpha\rfloor d\phi)\wedge{}^*d\phi-(e_\alpha\rfloor{}^*d\phi)\wedge\,d\phi\\&+m^2\,[(e_\alpha\rfloor\phi)\wedge{}^*\phi\,+\,(e_\alpha\rfloor{}^*\phi)\wedge\phi]\}\end{aligned} \tag{2.16}$$

is the energy–momentum current of the Proca field, and $\Sigma_\alpha^{(\mathrm{MAT})}$ is energy–momentum current of the additional matter field which satisfies also the corresponding Euler–Lagrange equations.

Thus, the triplet ansatz sector of a MAG theory coupled to a matter field has been reduced to the effective Einstein–Proca system of differential equations coupled to a matter field.

## III. STATIC BLACK HOLES IN MAG

In a recent work [9], we have investigated the gravitational field configuration corresponding to static spherically symmetric black holes in the context of the triplet ansatz sector of MAG, and we have proven a no-hair theorem for this specific case. It was shown that for the case of a massive Proca field ($m\neq 0$) in the presence of a static black hole, the effective

Proca field is trivial and the field equations reduce to the vacuum Einstein equations and, hence, the only static black hole is described by the Schwarzschild solution. Moreover, for a massless Proca field ($m = 0$), the equations reduce to the Einstein–Maxwell system and, therefore, the Reissner–Nordström solution is the only static black hole with non–degenerate horizon.

In addition, we have pointed out that for spherically symmetric static configurations of the triplet ansatz sector of MAG coupled to a Maxwell field, the only black hole solution allowed is the Reissner–Nordström one, because in this case the field equations are equivalent to an effective Einstein–Proca–Maxwell system. The question arises: Are these no–hair theorems valid also when the geometrical components of the triplet ansatz sector of MAG become (minimally) coupled to a different kind of matter fields?

### A. Matter fields with non–Abelian structure

In this section, we will show that our no–hair theorems are not valid in the presence of matter fields characterized by a non–Abelian gauge structure.

Most of the proofs of no–hair theorems are based upon the method first developed by Bekenstein [10] which consists on rearranging the field equations into a statement about the behavior of fields outside the event horizon. We have improved this method in our previous work [9] to prove the no–hair theorems for the triplet ansatz sector of MAG. Following [11] we will show that when an additional non–Abelian matter field is taken into account, the original argument for no–hair can be avoided.

Consider a set of arbitrary matter fields $\Psi_i$ in a gravitational background described by the effective Einstein–Proca system of MAG. The corresponding action becomes $S = \int (V_{\rm MAG} + \mathcal{L}_{\rm MAT}) d^4x$. After multiplication by $\Psi_i d^4x$ and integration, the Euler–Lagrange equations for the matter fields can be written as

$$\sum_i \int_\Omega \partial_\mu \left[ \Psi_i \frac{\partial \mathcal{L}_{\rm MAT}}{\partial(\partial_\mu \Psi_i)} \right] d^4x = \sum_i \int_\Omega \left[ \Psi_i \frac{\partial \mathcal{L}_{\rm MAT}}{\partial \Psi_i} + \partial_\mu(\Psi_i) \frac{\partial \mathcal{L}_{\rm MAT}}{\partial(\partial_\mu \Psi_i)} \right] d^4x \ . \tag{3.1}$$

Assuming that this coupled system admits black hole solutions, then the left–hand side of Eq. (3.1) can be expressed as a surface integral over the hypersurface $\partial\Omega$ which bounds the volume $\Omega$ exterior to the black hole. As has been shown by Bekenstein [10], this surface integral vanishes for static fields if the "norm" of the integrand

$$\sum_{i,j} g_{\mu\nu}\Psi_i\Psi_j \frac{\partial\mathcal{L}_{\rm MAT}}{\partial(\partial_\mu\Psi_i)}\frac{\partial\mathcal{L}_{\rm MAT}}{\partial(\partial_\nu\Psi_j)} \tag{3.2}$$

is finite on the horizon. Furthermore, if one can show that the integrand of the right hand side of Eq. (3.1) is either positive or negative definite, then the only solutions with finite enegy are those for which the integrand vanishes. This is the most used method to prove no–hair theorems in different theories (see Ref. [12] for a more recent approach).

Consider the Lagrangian for an Abelian Yang–Mills matter field: $\mathcal{L}_{\rm MAT} = \sqrt{-g}|F|^2 = \sqrt{-g}F_{\mu\nu}{}^a F^{\mu\nu}{}_a$ where $F_{\mu\nu}{}^a = 2\partial_{[\mu}A_{\nu]}{}^a$ and $a$ is the internal index. (The coupling constants are irrelevant for our analysis.) The calculation of the right–hand side of Eq. (3.1) yields $2\int_\Omega \sqrt{-g}|F|^2 d^4x$. If we consider a static field and, for the sake of simplicity without loss of generality, assume that the time component $A_t{}^a = 0$ , then $|F|^2 \geq 0$. This shows that the argument for no–hair can be applied in this case.

Consider now the Lagrangian for a non–Abelian Yang–Mills matter field. Due to the nonlinear terms of the field strength ($e$ is the gauge coupling constant)

$$F_{\mu\nu}{}^a = 2\partial_{[\mu}A_{\nu]}^a + e\epsilon^a{}_{bc}A_\mu{}^b A_\nu{}^c \ , \tag{3.3}$$

the right–hand side of Eq. (3.1) gives

$$2\sum_a \int_\Omega \sqrt{-g}\left[|F|^2 + e\epsilon_{abc}A_\mu{}^b A_\nu{}^c F^{\mu\nu\, a}\right] \ . \tag{3.4}$$

As in the Abelian case, one can show that $|F|^2 \geq 0$ for static fields. However, the second term in the integrand of Eq. (3.4) has not a definite sign, but depends on the particular solution for the exterior field of the black hole. This opens the possibility of avoiding the original no–hair argument. That is, the non–Abelian structure of the matter field can affect the statement about no–hair in the exterior of a static black hole. It is worth mentioning

that a similar argument can be used to show that an additional potential term in the matter Lagrangian of an Abelian Yang–Mills field can affect the no–hair statement in the same way as does the non–Abelian structure of the field by itself.

### B. Black holes with hair

In all theories in which black hole solutions with hair have been discovered, only the special case of static spherically symmetric spacetimes has been analyzed. For this reason, we will now investigate the triplet ansatz sector of MAG for this specific case. In the standard tensor notation, the field equations for the effective Einstein–Proca sector of MAG (2.13) can be written as follows

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \widetilde{\kappa} \left( \Sigma^{(\phi)}_{\mu\nu} + \Sigma^{\rm MAT}_{\mu\nu} \right) \ , \tag{3.5}$$

with

$$\Sigma^{(\phi)}_{\mu\nu} = H_{\mu}{}^{\lambda} H_{\nu\lambda} - \frac{1}{4} g_{\mu\nu} H_{\lambda\tau} H^{\lambda\tau} + m^2 \phi_\mu \phi_\nu - \frac{m^2}{2} g_{\mu\nu} \phi_\lambda \phi^\lambda \,, \tag{3.6}$$

where $R_{\mu\nu}$ is the Ricci tensor, $R$ is the curvature scalar, $H_{\mu\nu} \equiv 2\nabla_{[\mu}\phi_{\nu]}$ is the field strength of the *Abelian* Proca field $\phi_\mu$, and $\widetilde{\kappa} \equiv \kappa z_4 k_0^2/4\pi\rho a_0$. Moreover, the Proca field must satisfy the motion equation (2.14) which in tensor notation reads

$$\nabla_\nu H^{\nu\mu} = m^2 \phi^\mu \ . \tag{3.7}$$

Finally, the energy–stress tensor for the additional matter $\Sigma^{\rm MAT}_{\mu\nu}$ can be explicitly calculated from the corresponding matter Lagrangian.

We will consider asymptotically flat static spherically symmetric black hole spacetimes and write the corresponding coframe as

$$\vartheta^{\hat 0} = e^{-\delta} \, \mu^{1/2} d\, t \,, \quad \vartheta^{\hat 1} = \mu^{-1/2} \, d\, r \,, \quad \vartheta^{\hat 2} = r \, d\, \theta \,, \quad \vartheta^{\hat 3} = r \, \sin\theta \, d\, \varphi \,, \tag{3.8}$$

where $\delta$ and $\mu = 1 - 2M(r)/r$ are functions of $r$ only. The coframe is assumed to be *orthonormal* with the local Minkowski metric $o_{\alpha\beta} := \mathrm{diag}(-1,1,1,1) = o^{\alpha\beta}$. The condition

that the metric corresponding to the coframe (3.8) describes the gravitational field of a black hole implies that there exists a regular event horizon at a finite distance, say $r_H$, so $M(r_H) = r_H/2$, and $\delta(r_H)$ must be finite. On the other hand, asymptotic flatness requires that $\mu \longrightarrow 1$ and $\delta \longrightarrow 0$, at infinity. We also assume that the Proca field as well as the matter fields to be considered below respect the symmetries of the spacetime, i.e. they are static and spherically symmetric.

The Einstein's equations together with the equations of motion for the Proca and matter fields (2.13) form a dependent set as they are related by the Bianchi identities which in this case can be written in the form of a conservation law $\nabla_\mu \Sigma^{\mu\nu} = 0$. This conservation equation has only one non–trivial component (the $r$ component) which can be written as:

$$e^{\delta}(e^{-\delta}\Sigma^r_r)' = \frac{1}{2\mu r}[(\Sigma^t_t - \Sigma^r_r) + \mu(2\Sigma - 3(\Sigma^t_t + \Sigma^r_r))] \ . \tag{3.9}$$

Moreover, the field equations (3.5) for the coframe (3.8) yield

$$\mu' = \widetilde{\kappa}\, r\, \Sigma^t{}_t + \frac{1-\mu}{r}, \tag{3.10}$$

$$\delta' = \frac{\widetilde{\kappa}\, r}{2\,\mu}\,(\Sigma^t{}_t - \Sigma^r{}_r), \tag{3.11}$$

where the prime stands for differentiation with respect to the radial coordinate $r$, and $\Sigma_{\mu\nu} = \Sigma^{(\phi)}_{\mu\nu} + \Sigma^{\rm MAT}_{\mu\nu}$.

The set of equations (3.9)–(3.11) have been intensively analyzed in the literature for different theories and numerical solutions have been found that are characterized by the presence of non–Abelian and Higgs hair. These theories are: $SU(2)$ Yang–Mills, Skyrme, Yang–Mills–dilaton, Yang–Mills–Higgs and non–Abelian Proca. Because of the additivity of the energy–momentum tensor, hair will also exist in any theory which involves an arbitrary combination of the matter fields mentioned above. This is true also for any linear combination of energy–stress tensors in which at least one of them is characterized by the presence of hair. Accordingly, black hole hair will exist when any one of these matter fields is present in the gravitational field of a static spherically symmetric black hole described by the triplet ansatz sector of MAG.

### C. The no short hair conjecture in MAG

In a recent work [13], it was conjectured that *if a black hole has hair, then it cannot be shorter than the radius* $r_{\rm hair} = 3/2\sqrt{A/4\pi}$, *where* $A$ *is the horizon area.* In the case of a static spherically symmetric black hole the hair radius $r_{\rm hair} = 3/2 r_{\rm H}$, where $r_{\rm H}$ is the horizon radius. The hair radius defines around a black hole a hypersurface, called the "hairosphere", beyond which the hair can exist. Inside the hairosphere, hair is not allowed to exist. In this section we will show that this *no short hair conjecture* is valid for the special case of MAG under consideration in presence of the matter fields in which black hole hair has been discovered.

In all cases we will consider, the effective Einstein–Proca field equations of MAG (3.10) and (3.11) and the corresponding ansatz for the matter fields can be written as:

$$\mu' = \frac{1}{r}[1 - \mu + \alpha\,(K + U)], \qquad \delta' = \beta\, K\ , \tag{3.12}$$

where $\alpha, \beta$, $K$ and $U$ take particular values in each case. The strategy for showing the validity of the no short hair conjecture is the following. From the matter field equations and Eq. (3.12) it is possible to obtain a generic function [14] of the form $E \propto e^{-\delta}(K - U)$ which enters the conservation equation (3.9). If we demand the existence of a black hole solution in each case, we will show that $E$ must be negative on the horizon and positive semidefinite at infinity. Therefore, $E$ must be positive in some region (between the horizon and infinity) which is determined by the condition $3\mu > 1$. Finally, we show that this region corresponds to values of the radial coordinate $r$ outside the hairosphere. This proves the validity of the conjecture.

We now investigate all the particular cases in which black hole hair has been found. For the sake of simplicity, in each case we will quote for $\alpha$, $\beta$, $K$ and $U$ in Eq. (3.12) only the term corresponding to the additional matter field, dropping the term coming from the effective Proca field of MAG which does not allow the presence of hair [9]. Because of the additivity of the stress–energy tensor discussed above, this simplification does not affect the behavior of the generic function $E$.

$i$) The $SU(2)$ Yang-Mills [15] field for which the matter Lagrangian has the form

$$\mathcal{L}_{\rm MAT} = \mathcal{L}_{\rm YM} = -\frac{\sqrt{-g}}{16\pi f^2} F_{\mu\nu}{}^{a}\, F^{\mu\nu}{}_{a}, \tag{3.13}$$

where $F_{\mu\nu}{}^{a} = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + \epsilon^a{}_{bc} A_\mu{}^b\, A_\nu{}^c$ is the field strength for the gauge field $A_\mu{}^a$, and $f$ represents the gauge coupling constant. We use the static spherically symmetric ansatz for the potential

$$A = \sigma_a\, A_\mu{}^a\, dx^\mu = \sigma_1\, w\, d\,\theta + (\sigma_3\, \cot\theta + \sigma_2\, w) \sin\theta\, d\varphi, \tag{3.14}$$

where $\sigma_i$ $(i = 1, 2, 3)$ are the Pauli matrices and $w$ is a function of $r$ only. The field equations for this case may be written as in Eq. (3.12), with $K = \mu\, w'^2$, $U = (1-w^2)^2/(2r^2)$, $\alpha = -2/f^2$, and $\beta = -2/(f^2 \mu r)$. From the matter field equations we obtain

$$E' \equiv [r^2 e^{-\delta}(K - U)]' = r e^{-\delta}(3\mu - 1) w'^2\ . \tag{3.15}$$

From the expressions for $K$ and $U$ we see that $E$ is negative at the horizon because $K(r_H) = 0$, (since $\mu(r_H) = 0$), and $U(r_H) > 0$. On the other hand, the asymptotic flatness condition implies that $E \to 0$ as $r \to \infty$. Accordingly, $E$ must be an increasing function of $r$ in some intermediate region. It follows then that the right hand side of Eq. (3.15) must become positive at some point, i.e. we must have $3\mu > 1$.

$ii$) The Skyrme field with the matter Lagrangian [16]

$$\mathcal{L}_{\rm MAT} = \mathcal{L}_{\rm SK} = \sqrt{-g}\frac{f^2}{4}\, Tr(\nabla_\mu W \nabla^\mu W^{-1}) + \frac{\sqrt{-g}}{32\, e^2} Tr[(\nabla_\mu W)\, W^{-1}, (\nabla_\nu)\, W^{-1}]^2, \tag{3.16}$$

where $\nabla_\mu$ is the covariant derivative, $W$ is the $SU(2)$ chiral field, and $f^2$ and $e^2$ are the coupling constants. For the $SU(2)$ chiral field we use the *hedgehog* ansatz $W(r) = exp(\sigma \cdot \mathbf{r}\, F(r))$ where $\sigma$ are the Pauli matrices and $\mathbf{r}$ is a unit radial vector.

To write down the field equations we follow [16] and use the variables $\tilde{r} = e\, f\, r$, and $\tilde{m}(\tilde{r}) = e f m(r)$ so that the function $\mu$ defined above remains invariant. Dropping the tilde, the resulting equations are equivalent to Eq. (3.12) with $K = \mu\, [r^2/2 + \sin^2 F(r)] F'^2$, $U = \sin^2 F\, [1 + \sin^2 F\, /(2\, r^2)]$, $\alpha = -8\pi f^2$, and $\beta = -8\pi f^2/(\mu r)$. From the matter field equations we find

$$E' \equiv [e^{-\delta}(K-U)]' = -e^{-\delta}\left[r\mu F'^2 + \frac{1-\mu}{r\mu}K - 2r\left(1+\frac{U}{r^2}-\sqrt{1+2\frac{U}{r^2}}\right)\right]. \tag{3.17}$$

From the explicit expressions for $K$ and $U$ it follows that $E(r_H) < 0$, and since the asymptotic behavior of the field equations implies $F(r) \approx 1/r^2$ at infinity, we have that $E \to +0$ at infinity. Therefore, the right hand side of Eq. (3.17) must be positive in some region. Moreover, there must be a point where $E = 0$, i.e. $K = U$, and $E' > 0$. At this point, the right hand side of Eq. (3.17) becomes

$$-r\mu F'^2 - 2r\left(\sqrt{1+2\frac{K}{r^2}}-1\right) + \frac{K}{r\mu}(3\mu-1) > 0. \tag{3.18}$$

Since the first and second terms of the last equation are negative, we conclude that $3\mu > 1$ at this point. This is the same condition as in case $i$).

$iii$) In the case of $SU(2)$ Yang–Mills–dilaton field with an arbitrary (positive semidefinite) potential term $V(\phi)$ (which is expected to arise in superstrings inspired models [17]), the corresponding matter Lagrangian is given by [18]

$$\mathcal{L}_{\rm MAT} = \mathcal{L}_{\rm YMD} = \frac{\sqrt{-g}}{4\pi}\left(\frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi - \frac{1}{4f^2}e^{2\gamma\phi}F_{\mu\nu}{}^{a}\,F^{\mu\nu}{}_{a} - V(\phi)\right), \tag{3.19}$$

where $f$ is the gauge coupling constant, $\gamma$ is the dimensionless dilatonic coupling constant, and $F_{\mu\nu}{}^{a}$ is the SU(2) Yang–Mills field strength. The ansatz for the gauge field configuration is the same as that given in case $i$), and $\phi = \phi(r)$.

The corresponding field equations can be written in the generic form (3.12) with $K = K_1 + K_2$, where $K_1 = \mu\,\exp(2\gamma\phi)w'^2/f^2$, $K_2 = \mu r^2\phi'^2/2$ and $U = r^2V(\phi) + \exp(2\gamma\phi)(1-w^2)^2/(2f^2r^2)$, $\alpha = -2$, and $\beta = -2/(\mu r)$. Following the same procedure, from the matter field equations we find

$$E' \equiv [r^2e^{-\delta}(K-U)]' = re^{-\delta}\left[-2\,K_2 - 4r^2V(\phi) + (3\mu-1)\,\frac{K}{\mu}\right]. \tag{3.20}$$

The behavior of the generic function $E$ is as in the previous cases, and since the first and second terms of the right hand side of Eq. (3.20) are negative, we again find the condition $3\mu > 1$ in order to obtain asymptotically flat solutions.

*iv*) For a $SU(2)$ Yang–Mills–Higgs field the matter Lagrangian is given by [11]

$$\mathcal{L}_{\rm MAT} = \mathcal{L}_{\rm YMH} = -\frac{\sqrt{-g}}{4\pi}\left[\frac{1}{4f^2}F_{\mu\nu}{}^{a}\,F^{\mu\nu}{}_{a} + (D_\mu\Phi)^\dagger\,(D^\mu\Phi) + V(\Phi)\right], \tag{3.21}$$

where $D_\mu$ is the usual gauge–covariant derivative, $\Phi$ is a complex doublet Higgs field, and $F^{\mu\nu}{}_{a}$ is the $SU(2)$ Yang–Mills field given above. The arbitrary potential $V(\Phi)$ must be positive semidefinite. In this case, the ansatz for the Yang–Mills field is the same as before, and for the Higgs field we have

$$\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \varphi(r) \end{pmatrix}. \tag{3.22}$$

The field equations are equivalent to Eq. (3.12) with $K = K_1 + K_2$, where $K_1 = \mu r^2\varphi'^2/2$, and $K_2 = \mu w'^2/f^2$, $U = r^2V(\varphi) + (1-w^2)^2/(2f^2r^2) + (1+w)^2\varphi^2/4$, $\alpha = -2$, and $\beta = -2/(\mu r)$. Finally, the matter field equations lead to

$$E' \equiv [r^2e^{-\delta}(K-U)]' = re^{-\delta}\left[-2\,K_2 - 4r^2V(\varphi) - \frac{1}{2}(1+w)^2\varphi^2 + (3\mu-1)\,\frac{K}{\mu}\right]. \tag{3.23}$$

As in the previous cases, the required behavior of the function $E$ and the fact that the first three terms of the right hand side of Eq. (3.21) are negative lead to the condition $3\mu > 1$ for the region of interest.

*v*) In the case of a non–Abelian Proca field the matter Lagrangian is [11]

$$\mathcal{L}_{\rm MAT} = \mathcal{L}_{\rm NAP} = -\frac{\sqrt{-g}}{16\pi f^2}F_{\mu\nu}{}^{a}\,F^{\mu\nu}{}_{a} - \frac{\sqrt{-g}\,m^2}{32\pi}A^a_\mu A^\mu_a\,, \tag{3.24}$$

where $m$ is the mass parameter and the ansatz for the potential is as in the Yang–Mills case (3.14). Again, the field equations are given by Eq. (3.12) with $K = \mu w'^2$, $U = (1-w^2)^2/(2r^2) + m^2(1+w)^2$, $\alpha = -2/f^2$ and $\beta = -2/(f^2\mu r)$. On the other hand, from the matter field equations we obtain

$$E' \equiv [r^2e^{-\delta}(K-U)]' = re^{-\delta}\left[(3\mu-1)w'^2 - \frac{f^2m^2}{2}(1+w)^2\right]. \tag{3.25}$$

As in the previous cases, the required behavior of the generic function $E$ leads to the condition $3\mu > 1$.

In all cases presented here, there is a change in the behavior of the generic function $E$: It always starts at the horizon as a negative and decreasing function and needs to increase towards its asymptotic value. We have shown that this change always occurs beyond the point characterized by $3\mu > 1$. Since $\mu = 1 - 2M(r)/r$, the change occurs at the point $r_0 > 3M(r_0)$. On the other hand, $M(r)$ is an increasing function because from the general field equation (3.10) we have that $M' = -(\widetilde{\kappa}/2)r^2\Sigma^t{}_t = (\widetilde{\kappa}/2)r^2\rho_{\rm E} > 0$, where $\rho_{\rm E}$ is the energy density of the matter field which we suppose to be positive semidefinite in accordance with the weak energy condition. Being an increasing function, $M(r)$ reaches its minimum value on the horizon, where $\mu(r_{\rm H}) = 0$ and $M(r_{\rm H}) = r_{\rm H}/2$. Consequently, the turning point $r_0$ satisfies the inequality $r_0 > 3M(r_0) \geq 3M(r_{\rm H}) = 3r_{\rm H}/2$.

This result shows that the asymptotic behavior of the matter fields present in the gravitational field of a static spherically symmetric black hole can start only after the value of the radial coordinate $r$ is sufficiently large, and the lowest value determines the radius $r_{\rm hair}$ of the hairosphere. This proves the validity of the no short hair conjecture for the triplet ansatz sector of MAG in the presence of matter fields in which black hole hair has been found.

## IV. DISCUSSION

We have investigated the gravitational field of static spherically symmetric black holes described by the effective Einstein–Proca field of the triplet ansatz sector of MAG in the presence of matter. It was shown that when matter is represented by an Abelian Yang–Mills field, the no–hair theorems proven previously can be applied. On the other hand, if the matter possesses a non–Abelian structure or the corresponding Lagrangian contains an additional potential term, the arguments employed to prove the validity of no–hair theorems can be avoided due to the presence of an additional term in the general matter field equations.

In particular, it was shown that black hole hair exists in the system composed by the effective Einstein–Proca field of MAG and a $SU(2)$ Yang–Mills, Skyrme, Yang–Mills–dilaton,

Yang–Mills–Higgs or non–Abelian Proca field. Moreover, we have proved that in all these cases the no short hair conjecture is valid, that is, hair exists only outside a sphere of radius $r_{\rm hair} = 3r_{\rm H}/2$, where $r_{\rm H}$ is the horizon radius.

These results could be used to further investigate the physical significance of MAG. The no hair theorems proven in our previous work [9] show that the triplet ansatz sector of MAG in the presence of a spherically symmetric black hole is nothing more but Einstein's gravity. No new physics can be found in this sector because the no hair theorems prohibit the existence of more general solutions than the ones known in Einstein's gravity. However, the triplet ansatz sector is probably one the most simplest special cases of MAG. It could be that by slightly relaxing the triplet ansatz (2.10), one would obtain a more general effective system which could be still equivalent to Einstein's gravity coupled to a matter field. A first natural candidate could be the effective Einstein–non–Abelian–Proca field. In this case, as we have seen in this work, there exist solutions with black hole hair. The hair could then be directly related to some specific parts of the post–Riemannian structures of MAG. This research program, if realizable, could throw light on the physical significance of torsion and nonmetricity.

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## REFERENCES


[1] H. Weyl, *Naturwiss.* **19** (1931) 49.

[2] E. S. Fradkin and A. A. Tseytlin, *Phys. Lett.* **B158** (1985) 316; C. G. Callan, D. Friedan, E. J. Martinec and M. J. Perry, *Nucl. Phys.* **262** (1985) 593; D. Gross, *Phys. Rev. Lett.* **60** (1988) 1229; D. Gross and P. F. Mende, *Nucl. Phys.* **B303** (1988) 407 and *Phys. Lett.* **B197** (1987) 129.

[3] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne'eman, *Phys. Rep.* **258** (1995) 1. For computer algebra programs see also: J. Socorro, A. Macías, and F.W. Hehl, *Computer Physics Communications* **115** (1998) 264.

[4] F. W. Hehl and A. Macías, *Int. J. of Mod. Phys.* **D8** (1999) 339.

[5] Yu. N. Obukhov, E. J. Vlachynsky, W. Esser, and F. W. Hehl, *Phys. Rev.* **D56** (1997) 7769.

[6] R. Tucker, C. Wang: Non–Riemannian gravitational interactions. Talk given at *Mathematical Aspects of Theories of Gravitation*, Warsaw, Poland (1996). Banach Centre Publications Vol. **41** (Institute of Mathematics, Polish Academy of Sciences, Warsaw 1997). Los Alamos E-Print Archive: gr-qc/9608055.

[7] A. Macías, E. W. Mielke, and J. Socorro, *Class. Quantum Grav.* **15** (1998) 445.

[8] A. García, F. W. Hehl, C. Lämmerzahl, A. Macías, and J. Socorro, *Class. Quantum Grav.* **15** (1998) 1793. A. García, A. Macías, and J. Socorro, *Class. Quantum Grav.* **16** (1999) 93. A. Macías, C. Lämmerzahl, and A. García, *J. Math. Phys.* **41** (2000) 6369.

[9] E. Ayón–Beato, A. García, A. Macías and H. Quevedo, *Phys. Rev.* **D61** (2000) 084017.

[10] J. D. Bekenstein, *Phys. Rev. Lett.* **28** (1972) 452; *Phys. Rev.* **D5** (1972) 1239.

[11] B. R. Greene, S. D. Mathur, C. M. O'Neill, *Phys. Rev.* **D47** (1993) 2242.

[12] E. Ayón–Beato, *Phys. Rev.* **D62** (2000) 104004.

[13] D. Núñez, H. Quevedo and D. Sudarsky, *Phys. Rev. Lett.* **76** (1996) 571.

[14] D. Sudarsky, *Class. Quantum Grav.* **12** (1995) 579.

[15] J. A. Smoller, A. G. Wasserman and S. T. Yau, *Comm. Math. Phys.*, **154** (1993) 377.

[16] P. Bizon and T. Chamj, *Phys. Lett.* **B 297** (1992) 55; M. Heusler, S. Droz, and N. Straumann, *Phys. Lett.* **B268** (1991) 371; **B271** (1991) 61; **B258** (1992) 21.

[17] J. H. Horne and G. T. Horowitz, *Phys. Rev.* **D48** (1993) R5457.

[18] G. Lavrelashvili and D. Maison, *Nucl. Phys.* **B 410** (1993) 407.